HOMEWORK # 6 SOLUTIONS

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The objective of this homework is to test your understanding of the content of Module 6. Due date of the homework is: **Monday, March 7th, 2016, @ 18h00**.

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1. Using the Routh-Array method we discussed in Module 6 (in addition to the two special cases we discussed), determine whether the following system has any poles in the right half plane. Determine the number of these RHP poles.

$$H(s) = \frac{1}{s^4 + s^3 + 12s^2 + 12s + 36}.$$

Solutions: First, we construct the array:

$$\begin{array}{c||ccc} s^4 & 1 & 12 & 36 \\ s^3 & 1 & 12 & 0 \\ s^2 & 0 \rightarrow \approx \epsilon & 36 & \\ s^1 & \dfrac{12 \cdot \epsilon - 36}{\epsilon} & & \\ s^0 & 36 & & \end{array}$$

For $\epsilon > 0$, $\dfrac{12 \cdot \epsilon - 36}{\epsilon}$ is negative. Therefore, we have two sign changes in the TF which means we have two poles in the RHP.

2. For this CLTF,

$$H(s) = \frac{1}{s^3 + 7s^2 + 11s + (5+K)},$$

determine the range of K (which could include negative numbers, for this problem only) such that the system has no poles in the RHP. Consider all the cases.

Solutions: First, we construct the Routh array:

$$\begin{array}{c||cc} s^3 & 1 & 11 \\ s^2 & 7 & 5+K \\ s^1 & \dfrac{72-K}{7} & \\ s^0 & 5+K & \end{array}$$

Since it's required to have no poles in the RHP, we need no sign changes (Case 1) or zero entries (Case 2).

Case 1: assuming that we want only positive signs in the first column of the array, we would need $72 - K > 0$ **AND** $5 + K > 0$, or:

$$-5 < K < 72.$$

Case 2: assuming that we are permitting some entries to be zero, we can have $K = 72$. In this case, we obtain an updated array:

$$\begin{array}{c||cc} s^3 & 1 & 11 \\ s^2 & 7 & 77 \\ s^1 & 14 & \textbf{Here, we had to replace this row with an auxiliary polynomial} \\ s^0 & 77 & \end{array}$$

In this case, we have no sign changes, hence $K = 72$ is an acceptable solution. Similarly, we select $K = 5$, and reconstruct the Routh array again. For $K = 5$, we also obtain no sign changes, therefore the range for acceptable values for K is:

$$\boxed{-5 \leq K \leq 72.}$$

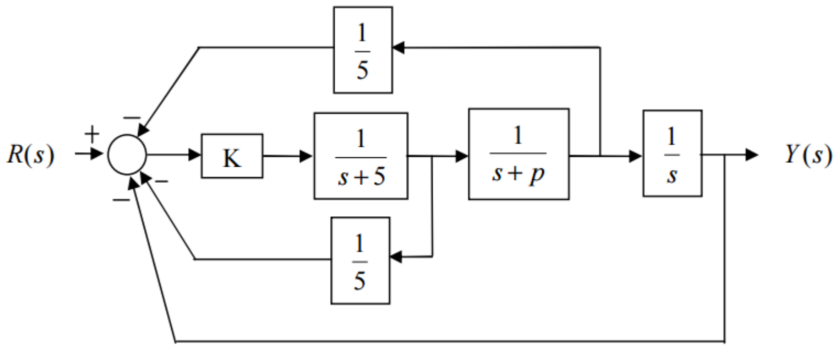


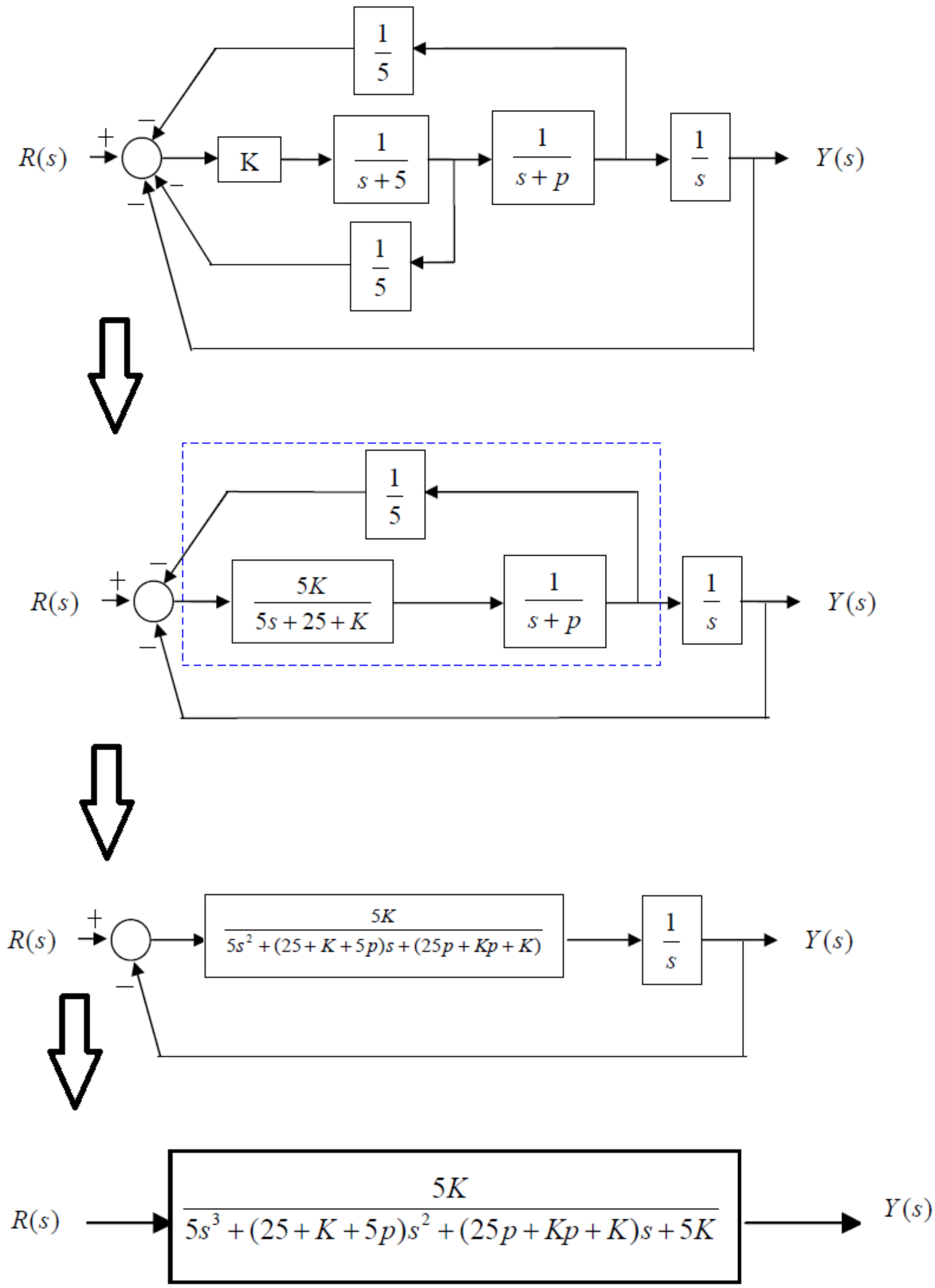
Figure 1: Spark ignition system — a block diagram representation.

3. The system shown in Figure 1 is a typical block diagram representation depicting a simplified spark ignition system of an engine. Two parameters of this system are the gain K and performance parameter p. The performance parameter can take two values: $p_1 = 0$ and $p_2 = 2$. The objective of this problem is to design a gain K to stabilize the initially unstable system. Answer the following questions.

   (a) Find the overall closed loop transfer function $\frac{Y(s)}{R(s)}$ in terms of p and K. You should end up with a typical TF with polynomials on the denominator and numerator. Your CLTF should be third order TF. Make sure that your answer is correct before you move to the next question.

   (b) Obtain a value (or values) for K that would make the CLTF stable for the two given values of p, **simultaneously**. In other words, your design should stabilize the system whether $p = 0$ or $p = 2$.
   Hint: you should end up with two Routh-Arrays in terms of p and K.

Solutions:



(a)

(b) Since we have two different values for p, we need to construct two Routh arrays corresponding to the two values.

Case 1: For $p = 0$, the Routh array is as follows:

$$\begin{array}{c||cc} s^3 & 5 & K \\ s^2 & 25+K & 5K \\ s^1 & \dfrac{K^2}{25+K} & \\ s^0 & 5K & \end{array}$$

Since our objective is to obtain a stable CLTF for $p = 0$, we want to make sure that there are no sign changes in the first column. Hence, we need:

$$25 + K > 0, \quad \frac{K^2}{25+K} > 0, \quad 5K > 0.$$

The solution for the above inequalities is $K > 0$. Note that we want the intersection of the three inequalities to hold, not one of them. This is satisfied via $K > 0$.

Case 2: For $p = 2$, the Routh array is as follows:

$$\begin{array}{c||cc} s^3 & 5 & 50+3K \\ s^2 & 35+K & 5K \\ s^1 & \dfrac{3K^2+80K+1750}{35+K} & \\ s^0 & 5K & \end{array}$$

Since our objective is to obtain a stable CLTF for $p = 2$, we want to make sure that there are no sign changes in the first column. Hence, we need:

$$35 + K > 0, \quad \frac{3K^2+80K+1750}{35+K} > 0, \quad 5K > 0.$$

The solution for the above inequalities is also $K > 0$. Similar to Case 1, we want the intersection of the three inequalities to hold, not one of them. This is satisfied via $K > 0$. You can easily check that by finding the sign of $3K^2 + 80K + 1750$.

Overall Solution: Combining the two conditions from Cases 1 and 2, the solution is:

$$\boxed{K > 0.}$$

4. For the unity feedback system in shown in Figure 2, the open-loop TF is given as follows:

$$G(s) = \frac{K(s+\alpha)}{s(s+\beta)},$$

where K, α and β are parameters that I want you to design. The design objectives are:

- Steady-state error $\frac{1}{10}$;
- Closed-loop poles are: $p_{1,2} = -1 \pm j$.

Find α, β and K such that the above design objectives are satisfied.

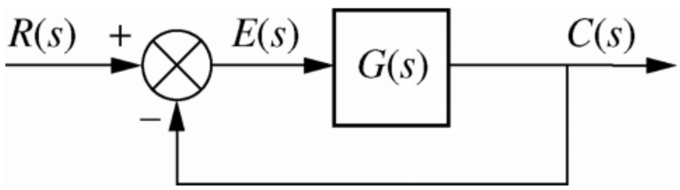


Figure 2: Unity feedback system.

Solutions: First, note that this is a Type 1 system as s^1 exists in the denominator. For Type 1 systems with ramp input, the steady-state error is equal to (class derivation and Module 06):

$$e_{ramp}(\infty) = \lim_{s\to 0} \frac{1}{sG(s)}.$$

According to the problem given, the steady state error is $1/10$, hence:

$$e_{ramp}(\infty) = \lim_{s\to 0} \frac{1}{sG(s)} = \frac{1}{10} = \frac{1}{s\frac{K(s+\alpha)}{s(s+\beta)}} = \frac{\beta}{\alpha K}.$$

Hence,

$$10 = \frac{\alpha K}{\beta} \Rightarrow \boxed{10\beta = \alpha K}.$$

Also, the problem given states that the poles of the CLTF are $-1 \pm j$. We can easily obtain the CLTF in terms of α, β, K:

$$\frac{C(s)}{R(s)} = \frac{K(s+\alpha)}{s^2 + (\beta + K)s + K\alpha}.$$

The poles of the CLTF are:

$$p_{1,2} = \frac{-\beta - K \pm \sqrt{\beta^2 + K^2 + 2\beta K - 4K\alpha}}{2}.$$

Since the real term in the poles is equal to -1, we have:

$$\frac{-\beta - K}{2} = -1 \Rightarrow \boxed{\beta = 2 - K}.$$

Furthermore, the complex part of the poles is equal to 1, hence we need the term under the square-root $(\beta^2 + K^2 + 2\beta K - 4K\alpha)$ to be equal to -4, since it will be square-rooted and then divided by two. Plugging in the boxed equations above (i.e., $\beta = 2 - K, 10\beta = \alpha K$), we obtain:

$$\boxed{K = \frac{72}{40} = 1.8 \Rightarrow \beta = 2 - K = 0.2, \Rightarrow \alpha = \frac{10\beta}{K} = 1.111.}$$

***Sanity check**:* plugging in these values for K, β, and α, we obtain the required steady-state error as well as the desired CLTF poles.