2017 IEEE International Conference on Robotics and Automation (ICRA)
Singapore, May 29 - June 3, 2017

Analytical Conditions for the Design of Variable Stiffness Mechanisms

Tze Hao Chong, Vincent Chalvet and David J. Braun*

Abstract— **This paper introduces an analytical approach for the design of variable stiffness mechanisms. The basis of this approach is a general model – representing the potential energy function and the physical constraints – covering the design space of variable stiffness mechanisms. Using this model, we present a systematic procedure to analytically define classes of variable stiffness mechanisms from first principles. Consequently, we identify mechanisms capable of infinite range stiffness modulation using bounded motor forces, and define the simplest mathematical model representing mechanisms in this class. A prototype mechanism consistent with this canonical model is designed, fabricated and experimentally tested. The experimental data are consistent with our theoretical predictions showing constant motor force independent of the output deflection and output stiffness when the mechanism is subject to external load.**

I. INTRODUCTION

First principles based design, through the use of mathematical conditions and algorithmic computation, has been a constant wish of designers since the introduction of computational kinematics [1]. However, traditional design is experience-based and heavily relies on intuition [2] since designing from mathematical conditions is nontrivial. Nevertheless there has been notable progress made in this area, in particular when it comes to input-output motion synthesis of rigid-body mechanisms [3] as well as optimal topological synthesis of compliant static structures [4]. Despite these, analytical approaches to the design of compliant actuators, series elastic actuators [5], variable stiffness actuators [6], and mechanisms capable of energy efficient stiffness modulation [7]–[10] require a general mathematical formalization. Even though some initiatives were established to provide insight to the design of these actuators [11], [12], a comprehensive theoretical framework which could enable classification, *a'priori* evaluation, and prediction of the physical features of these actuators, increasingly used in robot control applications [13], has yet to be developed.

In this paper we propose an analytical approach to characterize and design variable stiffness mechanisms. There are two main ingredients to this approach: one is the mathematical model of the potential energy function covering the design space of compliant mechanisms (Section II), while the other is given by user-defined design conditions to be satisfied by the actual mechanism (Section III). Imposition of these user-defined design conditions on the general model allows us to narrow down the design space to specific classes of variable stiffness mechanisms (Section IV).

This approach (1) provides the mathematical definition of the general class of variable stiffness mechanism (Section IV-A), useful for computational enumeration and automatic design of these mechanisms, and (2) leads to a distinctive sub-class of these mechanisms (Section IV-B), enabling infinite stiffness range using finite control forces [14], desirable when it comes to energy efficient stiffness modulation in robot control applications. Within the above mentioned sub-class of mechanisms, we identify a canonical variable stiffness mechanism. This canonical mechanism possesses: (a) linear relation between the input motor position and the output motion under constant loading condition, (b) inverse proportionality between the output stiffness and the input motor position, and (c) constant motor force required to change stiffness even under constant loading conditions. We present a prototype of such canonical mechanism (Section V) exhibiting all the above mentioned features, experimentally demonstrating the utility of the analytical design study presented in this paper.

II. MODEL OF VARIABLE STIFFNESS MECHANISMS

Variable stiffness actuators make use of two motor units to adjust their force-deflection characteristic Fig. 1. These actuators can provide output force, just like other non-compliant and non-variable stiffness actuators, but they can also provide concurrent control over the apparent output stiffness k_q (see Fig. 1). This is realized by a dedicated variable stiffness mechanism (Fig. 1 dashed lines). By construction, this mechanism makes use of an elastic element and a motor unit which manipulates the apparent stiffness k_q of the joint by actively changing the mechanism's non-linear kinematic structure.

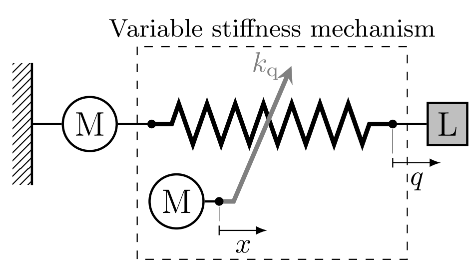

Fig. 1. Schematic representation of a variable stiffness actuator. The dashed lines label the variable stiffness mechanism. M's denote the motor units at the inputs, L denotes the external load at the output. x is the input position, q denotes the output position, while k_q is the output stiffness.

Tze Hao Chong, Vincent Chalvet and David J. Braun are with the Dynamics and Control Laboratory at the Singapore University of Technology and Design (SUTD).

This work has been funded by the SUTD-MIT International Design Center as part of the Energy-Efficient Compliant Actuator Designs project (IDG31400108), and in part, by the Changi General Hospital and SUTD HealthTech Innovation Fund Award (CGH-SUTD-2015-004). The authors gratefully acknowledge this support.

* E-mail: david_braun@sutd.edu.sg.

Variable stiffness mechanisms can be fully described by their potential energy function and their physical constraints [13]. The potential energy function is defined by the kinematic design and the way the compliant element couples the mechanism's input x to its output q. On the other hand, the constraints, limiting the motion, force and stiffness range of the mechanism, originate from the geometry of the design as well as the deformation and material limits of the compliant element. In the following we provide a dimensionless mathematical representation of these mechanisms, including both the potential energy function and the physical constraints.

(1) The potential energy function V is represented by the following series:

$$V(q,x) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} C_{m,n} q^m x^n, \tag{1}$$

where $(m,n) \in \mathbb{Z}^2$ denote the exponents of the polynomial basis functions $\{q^m x^n\}$ while $C_{m,n} \in \mathbb{R}$ are the constant weights in their linear combination. The positive exponents in the above series can be used to represent any analytic potential energy function according to Taylor's theorem [15]. The negative exponents extend this to a Laurent series representation which is capable of capturing a more general class of non-analytic functions [16]. This representation appears to be mathematically complete when it comes to potential energy functions of physically realizable mechanisms[1]. In relation to a practical mechanical design, the polynomial basis functions may be readily associated with the topology of the mechanism (i.e., type of the mechanism) while the constants $C_{m,n}$ are defined by the geometric dimensions and material properties of the design.

(2) The constraints imposed by the kinematic structure, as well as the deformation and material limitations, define the admissible space of the input and output positions. The minimalistic dimensionless representation of this admissible space is given by:

$$\mathcal{S} = \{(q,x) \in \mathbb{R}^2 : q \in [-1,1], x \in [0,1], |q| \leq q_{\text{max}}(x)\} \tag{2}$$

where $q_{\text{max}}(x)$ denotes the maximum deflection achievable at the output due to the deformation limit of the elastic element embedded in the mechanism. Similar to (1) we assume that this deformation limit can be represented by:

$$q_{\text{max}}(x) = \sum_{p=-\infty}^{+\infty} c_p x^p \tag{3}$$

where $c_p \in \mathbb{R}$ are constants. In relation to the mechanical design, the exponents in this series are associated with the topology of the design while the constants depend on the size, material properties of the particular design realization.

Together, (1) and (3) provide a general representation of variable stiffness mechanisms and is used here as an analytical model of a general mechanical design. Using this model we are now in position to physically characterize variable stiffness mechanisms. The quantities of interest are the force F_{q} and the stiffness k_{q} at the mechanism's output, and the force F_{x} at its input[2]. These three quantities are given by:

$$F_{\text{q}} = -\frac{\partial V(q,x)}{\partial q},\ k_{\text{q}} = \frac{\partial^2 V(q,x)}{\partial q^2},\ F_{\text{x}} = -\frac{\partial V(q,x)}{\partial x}. \tag{4}$$

Using the proposed representation, we can define one of the simplest potential energy functions by:

$$V(q,x) = C_{2,1} q^2 x. \tag{5}$$

According to this function, the force at the output is given by $F_{\text{q}} = -2C_{2,1}qx$. With unit inertial load, this model leads to a classical parametric oscillator: $\ddot{q} + (2C_{2,1}x)q = 0$ where the stiffness $k_{\text{q}} = 2C_{2,1}x$ is linearly related to the position of the stiffness adjusting motor x.

Another simple potential energy function of interest, which will be later encountered in this work, is given by:

$$V(q,x) = C_{2,-1}\frac{q^2}{x}. \tag{6}$$

As opposed to the above example, the force at the output of this mechanism is given by $F_{\text{q}} = -2C_{2,-1}qx^{-1}$. The stiffness of this mechanism $k_{\text{q}} = 2C_{2,-1}x^{-1}$ is inversely proportional to the position of the stiffness adjusting motor. It becomes clear that this second model, which represents a physically realizable design, requires the Laurent series representation in (1).

In this section we have introduced an analytical representation of variable stiffness mechanisms. This representation is general enough to capture mechanisms of different size, mechanical design and topological structure. In the next section we introduce a set of mathematical design conditions. These conditions will be used to find analytical models of variable stiffness mechanisms with user defined physical features.

III. Analytical design conditions

In this section we propose six analytical conditions representing the desired physical properties of variable stiffness mechanisms we wish to design. Each of these conditions will restrict the general mathematical representation of the potential energy function (1) and the constraint function (3). In particular, the restrictions will define the exponents (m,n,p) and the constants $C_{m,n}$ which should be present in the mathematical representation of a physically realizable variable stiffness mechanism. Concatenation of these conditions will enable us to analytically define important classes of variable stiffness mechanisms in the next section.

[1]The above Laurent series (1) can be used to represent a wide class of continuously differentiable potential energy functions. However, it is not able to capture discontinuous functions, therefore it may be possible to construct mechanisms that could not fit into this series representation on some level of abstraction. At present, we are not aware of such mechanisms.

[2]This input motor force is the one required to change the output stiffness of the variable stiffness mechanism.

A. Zero output force at the equilibrium

As in a typical spring, the output force F_q of the actuator should be zero at the equilibrium position $q = 0$. This condition should hold regardless of the mechanism's stiffness setting (i.e., regardless of the motor position x). This physical feature is captured by the following analytical condition:

$$F_\mathrm{q}(0,x) = -\frac{\partial V}{\partial q}(0,x) = 0, \quad \forall x \in [0,1]. \tag{7}$$

Imposition of this condition onto the model of the variable stiffness mechanism (1) leads to the following reduced set of basis polynomials:

$$\mathcal{B}_1 = \{q^m x^n\} \text{ where } (m,n) \in \left\{\mathbb{Z}^2 : m \geq 0, m \neq 1\right\}. \tag{8}$$

According to this condition, the potential energy function $V(q,x)$ could only have positive exponents of the output position (except the term that is linear with respect to this position). This condition already indicates that, when it comes to the output position q, a Taylor series representation of $V(q,x)$ is sufficient to model variable stiffness mechanisms.

B. Symmetric output force–displacement relation

In a typical spring positive output deflection creates negative force while negative deflection creates positive force. The underlying odd-symmetry of the force-displacement characteristic is one of the main features of variable stiffness mechanisms. The mathematical condition representing such symmetric force-displacement characteristic is given by:

$$F_\mathrm{q}(q,x) = -\frac{\partial V}{\partial q}(q,x) = \frac{\partial V}{\partial q}(-q,x) = -F_\mathrm{q}(-q,x), \quad \forall (q,x) \in \mathcal{S}. \tag{9}$$

Application of this condition to the model of the variable stiffness mechanism (1) leads to the following set of basis polynomials:

$$\mathcal{B}_2 = \{q^m x^n\} \text{ where } (m,n) \in \left\{\mathbb{Z}^2 : m = 2k, k \in \mathbb{Z}\right\}. \tag{10}$$

This result asserts that the polynomial basis function used in the potential energy function could only have even exponents in the output position q.

C. Controllable output stiffness

The essential role of a variable stiffness mechanism is to perform stiffness modulation. For this reason the apparent stiffness at the output k_q has to be controllable, throughout the workspace, by changing the input position x. This feature is represented by the following design condition:

$$\frac{\partial k_\mathrm{q}}{\partial x}(q,x) = \frac{\partial}{\partial x}\frac{\partial^2 V}{\partial q^2}(q,x) \neq 0, \quad \forall (q,x) \in \mathcal{S}. \tag{11}$$

Together with (8) this condition[3] asserts that it is necessary and sufficient to have at least one term in the potential energy function which is quadratic with respect to the output position and depends on the input position:

$$\exists n \neq 0 : C_{2,n} \neq 0. \tag{12}$$

This condition does not have any effect on the basis set, in the sense that it does not reduce the space of basis functions as the previous two conditions did. Instead, this condition defines a minimal requirement for the mathematical representation of variable stiffness mechanisms. In fact, this condition is alone a necessary and sufficient condition for the design of variable stiffness mechanisms. The two examples introduced in the previous section i.e., (5) and (6) present mechanisms that satisfy this condition. This condition is also satisfied by all other variable stiffness mechanisms we know.

The first three conditions are respected by any variable stiffness mechanism having symmetric force-deflection characteristic. In the following we introduce three additional design conditions, enabling large range stiffness modulation with low motor force. This design feature is desirable when the aim is to achieve intrinsically low power stiffness modulation in practical application [9], [14], [17].

D. Zero input force at the equilibrium position

Previous studies have demonstrated the possibility to design intrinsically energy efficient variable stiffness actuators [7], [8], [11]. In these designs the force generated by the elastic element onto the stiffness modulating motor remains zero at the equilibrium position of the actuator. This design feature can be imposed by the following mathematical condition:

$$F_\mathrm{x}(0,x) = -\frac{\partial V}{\partial x}(0,x) = 0, \quad \forall x \in [0,1]. \tag{13}$$

Imposition of this condition onto the model of the mechanism (1) leads to the following basis polynomials:

$$\mathcal{B}_4 = \{q^m x^n\} \text{ where } (m,n) \in \left\{\mathbb{Z}^2 : m > 0\right\} \cup \left\{\mathbb{Z}^2 : n = 0\right\}. \tag{14}$$

Accordingly, the basis functions representing the potential energy of the mechanism could come from two sets. One of these sets only contains polynomials with strictly positive exponents of the output position, while the other one can contain terms that have arbitrary power of the output position but without the presence of the input position.

E. Large output stiffness range

The range of output stiffness is one of the main characteristics of variable stiffness mechanisms. One way to achieve large stiffness range is to design the mechanism in such a way that it can reach infinite stiffness i.e., capable of emulating a rigid joint. The corresponding condition is given by:

$$\max_{(q,x)\in\mathcal{S}} k_\mathrm{q}(q,x) = \max_{(q,x)\in\mathcal{S}} \frac{\partial^2 V}{\partial q^2}(q,x) = \infty. \tag{15}$$

Imposition of this constraint on the model of the mechanism (1) leads to the following existence condition:

$$\exists (m,n) \in \left\{\mathbb{Z}^2 : m \neq 0, m \neq 1, n < 0\right\} : C_{m,n} \neq 0. \tag{16}$$

[3] This condition must be valid at the equilibrium position ($q = 0$) but alone it does not guarantee controllability of the stiffness at the equilibrium position when $m < 0$, since in that case $\partial k_\mathrm{q}/\partial x|_{q=0} \to \infty$. However (8) excludes negative exponents in the output position and as such it eliminates this special case.

According to this condition, it is necessary and sufficient to have at least one term, in the series expansion of the potential energy function, which is inversely proportional to the input position and has non-linear dependency on the output position. Clearly, this will make the stiffness tend to infinity as the input position tends to zero: $\lim_{x\to 0} k_{\text{q}} = \infty$.

F. Low input motor force

While the stiffness tends to infinity, the input motor force could also reach high – theoretically infinite – values even if the mechanism is operated within its admissible workspace (2). In order to ensure that the mechanism is not subject to such an undesirable feature, one may impose the following inequality constraint:

$$\max_{(q,x)\in\mathcal{S}} F_{\text{x}}(q,x) < \infty. \tag{17}$$

Assuming that p is positive[4], this inequality defines the polynomial basis functions in the constraint (3):

$$\mathcal{B}_6 = \{x^p\} \text{ where } p \in \{\mathbb{Z} : n = 0\} \cup \{\mathbb{Z} : m p_{\text{min}} \geq 1 - n\}, \\ \forall (m,n) \in \left\{\mathbb{Z}^2 : C_{m,n} \neq 0\right\} \text{ and } p_{\text{min}} = \min_{p\in\{\mathbb{Z}:c_p\neq 0\}} p. \tag{18}$$

Depending on the terms in the potential energy function $\{q^m x^n\}$, the series representation of the constraint (3) can either be arbitrary when $n = 0$ or can only contain terms $\{x^p\}$ with integer exponents equal or larger than $p \geq \frac{1-n}{m}$.

While the previous design constraints define the basis of the potential energy function, the last constraint defines the basis functions representing the physical limitation of the compliant mechanism. The coupling between the elastic properties of the actuator and its physical limitations becomes apparent through this last condition. It turns out that this coupling plays a key role in the design of mechanisms capable of large range stiffness modulation using low control forces. This will be more directly shown in the next section.

IV. Analytical design

In this section we define classes of variable stiffness mechanisms which satisfy the analytical conditions stated above. We do this by simultaneously imposing the above conditions (8), (10), (12), (14), (16) and (18) on the original model (1) and (3) introduced in Section II. We use this approach to define: (A) a general class of variable stiffness mechanisms which covers mechanisms with symmetric output force-displacement characteristic, (B) a subclass of these mechanisms which enable infinite range stiffness modulation using finite input forces, and (C) two types of mechanisms within the above sub-class; one enabling infinite stiffness setting with nonzero but bounded input force, while the other enabling the same behaviour with zero input force.

[4]This ensures that the output position tends to zero when the stiffness is infinite.

A. Variable stiffness mechanisms

The first three conditions (8), (10) and (12), introduced in Sections III-A–III-C, are conditions satisfied by any variable stiffness mechanism having symmetric force-deflection characteristic. These conditions lead to the following polynomial basis functions:

$$\mathcal{B}_{\text{V}}^{\text{vsm}} = \mathcal{B}_1 \cap \mathcal{B}_2 = \{q^m x^n\} \\ \text{where } (m,n) \in \left\{\mathbb{Z}^2 : m = 2k, k \in \mathbb{N}\right\} \tag{19}$$

and an additional existence condition:

$$\exists n \neq 0 : C_{2,n} \neq 0. \tag{20}$$

According to (19) and (20), a potential energy function of variable stiffness mechanisms composed of several low order terms, can be given by:

$$V(q,x) = C_{0,0} + C_{0,1}x + C_{2,1}q^2 x. \tag{21}$$

Here negative exponents in the output position q cannot be present in (21) but negative exponents in the input position x can be present, according to the above two conditions.

B. Definition of the new class of mechanism

In the previous subsection we mathematically defined a general class of variable stiffness mechanisms. Here we aim to find a sub-class of these mechanisms characterized by infinite range stiffness modulation with finite input forces [14]. In order to mathematically define this sub-class, we impose all six conditions, presented in the previous section, on the general model (1)–(3). These conditions lead to the following basis functions of the potential energy:

$$\mathcal{B}_{\text{V}}^{0} = \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_4 = \{q^m x^n\}, \\ \text{where } (m,n) \in \left\{\mathbb{Z}^2 : m = 2k, k \in \mathbb{N}, k \neq 0\right\} \cup (0,0) \tag{22}$$

together with the additional necessary conditions:

$$\exists n \neq 0 : C_{2,n} \neq 0, \\ \exists (m,n) \in \left\{\mathbb{Z}^2 : n < 0\right\} : C_{m,n} \neq 0. \tag{23}$$

Furthermore, the set of basis functions representing the physical constraint is given by:

$$\mathcal{B}_{\text{q}}^{0} = \{x^p\}, \text{ where } p \in \left\{\mathbb{Z} : p_{\text{min}} \geq \frac{1-n}{m}\right\}, \\ \forall (m,n) \in \left\{\mathbb{Z}^2 : C_{m,n} \neq 0\right\} \text{ and } p_{\text{min}} = \min_{p\in\{\mathbb{Z}:c_p\neq 0\}} p. \tag{24}$$

Using these conditions we can now define a canonical variable stiffness mechanism. The model of this mechanism is defined by the lowest order and least number of basis terms which respect the conditions (22)–(24) presented above:

$$V(q,x) = C_{2,-1}\frac{q^2}{x} \quad \text{and} \quad q_{\max}(x) = c_1 x. \tag{25}$$

Compared to (21), where $V(q,x)$ does not necessarily contain terms with negative exponents in x, here at least one such term must be present. Also, while the basis functions representing the general class of variable stiffness mechanisms (19)-(20) do not define the constraint (3), here the constraint is defined.

C. Two distinct types of variable stiffness mechanisms

One important characteristic of this new class of mechanisms is the bounded input force F_x, as defined by (17). However, even within all mechanisms that are designed to possess bounded input force, two distinct types of mechanisms can be identified.

In order to show this, let us examine the behaviour of the input force on the boundary of the feasible set $q = q_{max}(x)$ close to the point where the mechanism displays infinite stiffness (i.e., $x \to 0$):

$$F_x(q_{max}(x), x)|_{x \approx 0} \approx nC_{m,n} c_{p_{min}}^m x^{mp_{min}+n-1}$$
$$\text{where } p_{min}m + n - 1 \geq 0 \ \ \forall (m, n) \in \left\{\mathbb{Z}^2 : C_{m,n} \neq 0\right\}. \quad (26)$$

The separation between the two different types of mechanisms is decided by the exponent in the above asymptotic expansion. In particular: (1) if a triplet (m, n, p_{min}) exists such that $mp_{min} + n - 1 = 0$, the input force will tend to a non-zero constant value $F_x(q_{max}(x), x)|_{x \to 0} = nC_{m,n} c_{p_{min}}^m$ as the stiffness tends to infinity, on the other hand (2) if $mp_{min} + n - 1 > 0$ then the input force will depend on x, and will tend to zero as the stiffness tends to infinity $F_x(q_{max}(x), x)|_{x \to 0} = 0$. This second type of mechanisms may approach infinite stiffness while no load is felt by the stiffness tuning motor.

There has been previous designs that belong to the first type of mechanisms. Examples of these designs are the variable stiffness torsion-cylinder mechanisms [18] – where $V \propto q^2/x$ and $q_{max}(x) \propto x$ – and the variable stiffness leaf-spring mechanism [9] – where $V \propto q^2/x^3$ and $q_{max}(x) \propto x^2$. However, we are not aware of previous designs which belong to the mechanisms of the second type. Nonetheless the above analytical relation (26) may shed light on how to design one such variable stiffness mechanism.

V. Implementation of the Canonical Mechanism

In this section we present a practical realization of the above defined canonical mechanism (25). In addition, we report experimental data verifying the previously defined analytical design features of this mechanism.

A. Variable stiffness mechanism

Figure 2a shows a canonical variable stiffness mechanism. In our realization[5], this device is composed by a torsional leaf-spring, a drive train which modulates the active length of the spring, and an output shaft coupled to the spring. The drive train changes the active length of the spring via a linearly actuated slider. This changes the stiffness at the output shaft.

The potential energy function and the deformation limit at the output shaft on this variable stiffness mechanism are given by:

$$V(q, x) = \frac{kah^3G}{2}\frac{q^2}{x} \quad \text{and} \quad q_{max}(x) = \frac{\sigma_y}{hG}x \qquad (27)$$

[5]The physical realization of this mechanism is not unique i.e., there are different ways to realize a canonical mechanism as defined by (25). In this section we present one such realization.

where[6] $q \in [-q_{max}, q_{max}]$ is the output position, $x \in [0, L]$ is the input position, L is the total length of the spring, G is the modulus of torsional rigidity, a and h are the width and the height of the torsional leaf-spring, $k \approx 1/3$ (for $a/h > 10$) is a geometric correction factor of a torsionally loaded rectangular leaf-spring [19] while σ_y is the yield strength of the spring.

In this mechanism, the motor position input x is directly related to the effective length of the spring. This leads to the previously identified inverse relation between the length of the spring and the output stiffness:

$$k_q = kah^3G\frac{1}{x}. \qquad (28)$$

Also, the maximum deflection at the output link $q_{max}(x)$ depends linearly on the motor position (27). Accordingly, the present design exhibits both of the characteristics of the above defined canonical variable stiffness design (25).

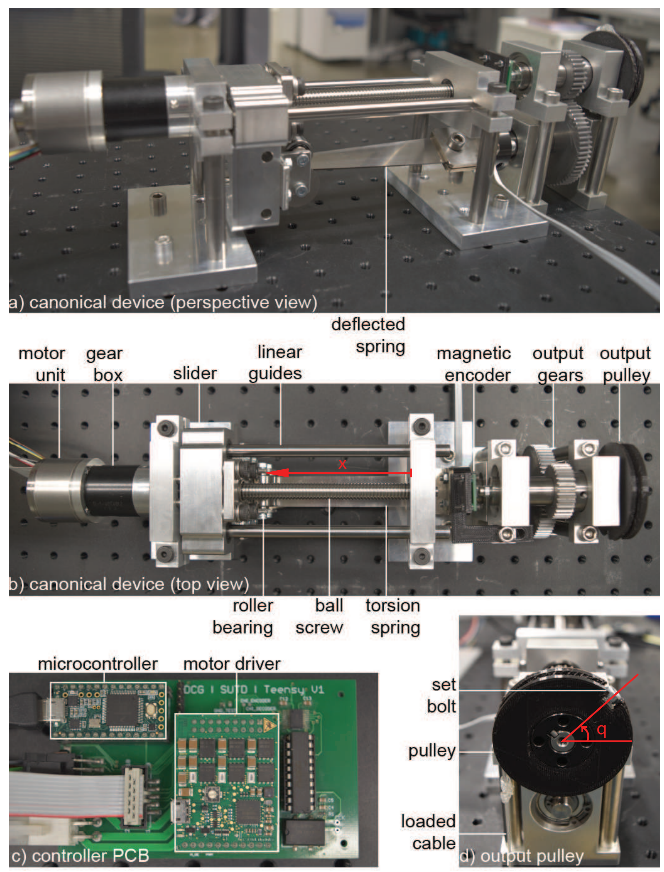

Fig. 2. Variable stiffness mechanism. From top to bottom, the figure shows: a,b) the device, c) the electronic control interface and d) the output pulley.

Control over the output stiffness, or more directly, the effective length of the torsional leaf-spring (20mm wide, 1mm thick full hard 301 stainless steel leaf) is realized by a Maxon ball-screw (10mm x 2mm pitch) driven by a brushless DC motor (Maxon EC-i 50W) coupled to a back-drivable planetary gearbox (4.8:1 gear ratio). The active length of the

[6]In this model q and x represent quantities with physical dimensions.

spring is determined by a quadrature encoder. During the experiments, the supply current is measured. This current measurement is used to estimate the input forces required to change the length of the torsional leaf-spring. The spring is connected to the output shaft via a 2:1 gear reduction. The angular deflection of the output shaft is determined by a rotary encoder. Closed-loop motor position control is implemented using the signal provided by the motor encoder. The motor is driven by a Maxon ESCON 50V/5A 4-Q driver while the entire system is operated at 500Hz by a micro-controller.

B. Experiment

In this section we present the experimental data characterizing the relation between the output stiffness and the input motor force on this variable stiffness mechanism. The data will be used to evaluate two design conditions, namely one that asserts high stiffness range (15) and another which indicates finite motor force irrespective of the stiffness setting (17).

At the start of the experiment, the mechanism was set to its minimum stiffness which corresponds to the largest active length of the spring $x = L = 90$ mm. The motor was driven with low constant speeds $\dot{x} \approx 0.002$ m/s until the active length of the spring reached its minimum length while the output stiffness reached its maximum value. During this quasi-static stiffness change, the output shaft was loaded such that a constant torque of $M = 2.25$ Nm was exerted on the spring i.e., the mechanism is operated away from its equilibrium configuration. The input motor force required for stiffness modulation F_{x} was found directly through the power supply voltage and current used to drive the motor at constant speed.

Figure 3 shows the measured output deflection q, the output stiffness k_{q}, and the input motor force F_{x} as functions of the input position x under constant external torque (i.e., $M = 2.25$ Nm). At the start of the experiment the stiffness was at its minimum value (the input position was set to its maximum value $x = 90$ mm). Throughout the experiment, the deflection of the spring decreased in a linear fashion $q \propto x$ due to the increase in stiffness, similar to the analytical model (Fig. 3a). As the length of the spring decreased from its maximum to its minimum, the stiffness increased as predicted by the theoretical model (Fig. 3b). During this transition, the input force is not only finite but it is nearly constant as predicted by the theoretical model (Fig. 3c), albeit the experimental value was found to be larger than the theoretical one. The performance of the device corroborates well with the analytical design conditions, showing high stiffness range (15) and finite input force (17). The prototype adequately represents the canonical variable stiffness mechanism.

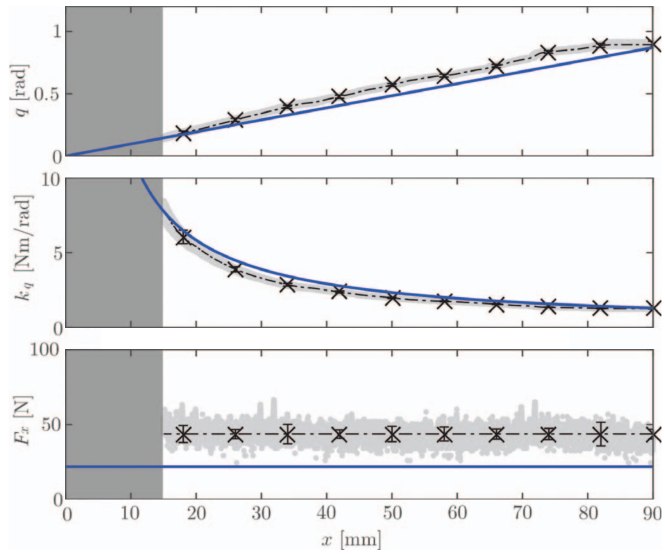

Fig. 3. Experimental data. a) Deflection angle versus input position. b) Stiffness versus input position. c) Input force versus input position. The data is collected throughout ten experiments (gray). The mean of the data is shown with dashed black lines. The error bars indicate two standard deviation. The prediction of the model (solid blue lines) is shown in the same plots. Due to the particular design realization the minimum value of x is 15 mm on this prototype.

VI. Conclusion

This paper introduces an analytical approach for the design of variable stiffness mechanisms. The basis of this approach is a general model – representing the potential energy function and the physical constraints – covering the design space of variable stiffness mechanisms. Using this model, we present a systematic procedure to define classes of variable stiffness mechanisms from first principles. The proposed approach may help addressing non-intuitive design problems; those that had been long identified challenging to treat using experience based approaches.

References

[1] F. Freudenstein and G. N. Sandor, "Synthesis of path generating mechanisms by means of a programmed digital computer," *ASME Journal of Engineering for Industry*, vol. 81, no. 1, pp. 159–168, 1959.

[2] E. S. Ferguson, "The mind's eye: Nonverbal thought in technology," *Science*, vol. 197, no. 4306, pp. 827–836, 1977.

[3] J. M. McCarthy and G. S. Soh, *Geometric design of linkages*, vol. 11. Springer Science & Business Media, 2010.

[4] L. L. Howell, *Compliant mechanisms*. John Wiley & Sons, 2001.

[5] G. Pratt and M. Williamson, "Series elastic actuators," in *IEEE/RSJ International Conference on Intelligent Robots and Systems*, vol. 1, (Pittsburg, PA), pp. 399–406, 1995.

[6] R. van Ham, T. Sugar, B. Vanderborght, K. Hollander, and D. Lefeber, "Compliant actuator designs," *IEEE Robotics & Automation Magazine*, vol. 16, no. 3, pp. 81–94, 2009.

[7] A. Jafari, N. G. Tsagarakis, and D. G. Caldwell, "A novel intrinsically energy effcient actuator with adjustable stiffness (awas)," *IEEE/ASME Transactions on Mechatronics*, vol. 18, no. 1, pp. 355–365, 2013.

[8] A. Jafari, N. G. Tsagarakis, I. Sardellitti, and D. G. Caldwell, "A new actuator with adjustable stiffness based on a variable ratio lever mechanism," *IEEE/ASME Transactions on Mechatronics*, vol. 19, no. 1, pp. 55–63, 2014.

[9] D. J. Braun, S. Apte, O. Adiyatov, A. Dahiya, and N. Hogan, "Compliant actuation for energy efficient impedance modulation," in *IEEE International Conference on Robotics and Automation*, (Stockholm, SE), pp. 636–641, 2016.

[10] A. Dahiya and D. J. Braun, "Efficiently tuneable positive-negative stiffness actuator," in *IEEE International Conference on Robotics and Automation*, (Singapore), May 2017.

[11] L. C. Visser, R. Carloni, and S. Stramigioli, "Energy-effcient variable stiffness actuators," *IEEE Transactions on Robotics*, vol. 27, no. 5, pp. 865–875, 2011.

[12] R. Carloni, L. C. Visser, and S. Stramigioli, "Variable stiffness actuators: A port-based power-flow analysis," *IEEE Transactions on Robotics*, vol. 28, no. 1, pp. 1–11, 2012.
[13] D. J. Braun, F. Petit, F. Huber, S. Haddadin, P. van der Smagt, A. Albu-Schaffer, and S. Vijayakumar, "Robots driven by compliant actuators: Optimal control under actuation constraints," *IEEE Transactions on Robotics*, vol. 29, no. 5, pp. 1085–1101, 2013.
[14] V. Chalvet and D. J. Braun, "Criterion for the design of low power variable stiffness mechanisms," *IEEE Transactions on Robotics*, 2017 (accepted).
[15] W. Rudin, *Principles of Mathematical Analysis*. New York, USA: McGraw-Hill, Inc., 3 ed., 1953.
[16] J. Van Iseghem and P. R. Graves-Morris, "Approximation of a function given by its Laurent series," *Numerical Algorithms*, vol. 11, no. 1, pp. 339–351, 1996.
[17] J. W. Hurst and A. A. Rizzi, "Series compliance for an efficient running gait," *IEEE Robotics & Automation Magazine*, vol. 15, no. 3, pp. 42–51, 2008.
[18] J. Schuy, P. Beckerle, J. Faber, J. Wojtusch, S. Rinderknecht, and O. von Stryk, "Dimensioning and evaluation of the elastic element in a variable torsion stiffness actuator," in *IEEE/ASME International Conference on Advanced Intelligent Mechatronics*, (Wollongong, AU), pp. 1786–1791, July 2013.
[19] S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*. New York, USA: McGraw-Hill Book Company, Inc., 1970.