

**Homework Instructions:**

1. Type your solutions in the L<sup>A</sup>T<sub>E</sub>X homework template file. Otherwise, you can use any other typesetting tool or you can provide handwritten solutions, assuming everything is clear.
2. **Due date:** Thursday, October, 8th, @ 5:00pm on Blackboard, **AND** drop off a copy of your solutions (slip it under the office door if I'm away).
3. **Collaboration policy:** you can collaborate with your classmates, under the assumption that everyone is required to write their own solutions. If you choose to collaborate with anyone, list their name(s).
4. You don't show your work  $\Rightarrow$  You don't get credit.
5. Solutions that are unclear won't be graded.
6. Before you start with this homework assignment, make sure that you have grasped the content of Module 05.

## Problem 1 — Cost of an Infinite Horizon LQR

Prove that the total cost of the CT, LTI infinite horizon LQR problem, given by:

$$\begin{aligned} \text{minimize } J &= \int_0^\infty \|y(t)\|^2 dt \\ \text{subject to} &\quad \dot{x}(t) = Ax(t) \\ &\quad y(t) = Cx(t) \end{aligned}$$

is  $J = x_0^\top P x_0$  where  $P$  is the solution to the steady-state Riccati equation, given in Module 05, and  $x(0)$  is the vector of initial state conditions.

*Hint:* Write the cost function as a quadratic cost function in terms of  $x(t)$  and then relate to CARE.

## Problem 2 — Infinite Horizon LQR

Compute  $J = \int_0^\infty x^\top \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} x dt$ , given that the system dynamics are given by:

$$\dot{x}(t) = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} x(t),$$

where  $x(0) = [0 \ 1]^\top$ . You are supposed to solve the problem analytically using two different methods of your choice (CARE is one of them). You are not supposed to use any programming tool. You should also use the result from Problem 1.

Verify your solutions using MATLAB. Show your code.

## Problem 3 — Two Point Boundary Value Problem

In this problem, we will learn about optimal control solutions for a two point boundary value problem (TPBVP) — an optimal control problem where terminal state conditions are pre-specified. You should do research on how to solve TPBVP with fixed final and initial states.

For example, you might find Example 6-1 in <http://goo.gl/CUIwPl> useful, as it includes an example on solving TPBVP. Also, read the *LQR Variational Solution* section in the linked PDF.

After reading the linked PDF and going through the example, find the optimal control trajectory,  $u^*(t) = [u_1^*(t) \ u_2^*(t)]^\top$ , that minimizes this performance index:

$$J = \frac{1}{2} \int_0^1 \|u(t)\|^2 dt,$$

subject to:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, \quad x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using MATLAB, plot your optimal control, performance index, J, and the corresponding state-trajectory.

## Problem 4 — Optimal Control for a Nonlinear System

Using the HJB-equation, and a candidate quadratic function  $V(t, x)$ , find the optimal control action that minimizes this performance index:

$$J = (x(1))^2 + \int_0^1 x^2(t)u^2(t) dt,$$

where the dynamics of the nonlinear system are given by:

$$\dot{x}(t) = x(t)u(t), \quad x(0) = 1.$$

You'll have to follow the following steps:

1. Construct the Hamiltonian.
2. Obtain the optimal control.
3. Apply property 1 of any value function, i.e.,  $\frac{\partial V}{\partial x} = \lambda^*(x, t)$ .
4. Substitute this optimal control and multiplier into the HJB equation.
5. Formulate a candidate quadratic value function.
6. Use this candidate in the HJB and obtain an ODE that relates to the value function.
7. Obtain the optimal control.
8. Finally, plot the optimal control, cost function, and the state trajectory (you can either use the ode solver on MATLAB).

## Problem 5 — Discrete LQR Solution + Lagrangian

The discrete dynamics of an LTI system is given by:

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \dots, N-1.$$

Consider the optimal control problem of finding optimal control sequence,  $u_0^*, \dots, u_{N-1}^*$ , given:

- Specific initial and final conditions:  $x_0$  and  $x_N$  are fixed, and

- Cost index:  $J = \frac{1}{2} \sum_{k=0}^{N-1} u_k^\top R u_k$

The objective of this problem is to transform the optimal control problem to a quadratic, static optimization problem subject to linear equality constraints.

Answer the following questions:

1. Write the cost function  $J$  as a quadratic cost function in  $u = [u_0 \ u_1 \ \dots \ u_{N-1}]^\top$ , where you should determine the quadratic cost-matrix—it should be diagonal.
2. Given an initial and final fixed states, write the dynamics of the system as  $A_u u = b_u$ , where  $A_u$  and  $b_u$  should be determined in terms of  $A, B, x_0, x_{N-1}$ .
3. Formulate the optimal control problem as a quadratic program with linear equality constraints.
4. Construct the Lagrangian of the transformed optimization problem.
5. What is the optimal  $u^*$ ? You have to solve a KKT-like problem for multipliers and control.

## Problem 6 — Principle of Optimality and DP

Consider the following discrete-time LQR problem:

$$\begin{aligned} & \text{minimize} \quad J = (x_2 - 10)^2 + \frac{1}{2} \sum_{k=0}^1 (x_k^2 + u_k^2) \\ & \text{subject to} \quad x_{k+1} = 2x_k - 3u_k \\ & \qquad \qquad \qquad x(0) = 4. \end{aligned}$$

The final state can be anything, i.e., it is free, not fixed. The objective of this problem is to solve the above optimal control problem by invoking the Principle of Optimality and Dynamic Programming, similar to the example on Slide 13 of Module 05.

Obtain the optimal control sequence and the corresponding state trajectories.